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# EFFICIENT ANALYTICAL AND NUMERICAL SOLUTIONS OF HAISSINSKI EQUATION

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Two possible techniques for solving the Haissinski equation are discussed. The first one applies to general impedance (wake) functions, and consist in recasting a perturbative solution into a Pade' approximant. The second applies to impedance functions having two (or more generally $n$) poles, and provides a numerical solution whose computation is considerably faster than obtainable by the standard Bane-Wilson algorithm. The proposed techniques are tested against available exact and numerical solutions.

KEY WORDS: Electron Storage Rings, Longitudinal Equilibrium Distribution, Wake Field Effects

## 1 INTRODUCTION

The Haissinski (or Potential Well Distortion, henceforth PWD) equation[1] is a non linear integral equation describing, in a suitable phase space, the evolution of the longitudinal equilibrium particle density distribution for an electron bunch traveling in a storage ring.

The nonlinear term contains the so called *wake-potential*, which is generated by the electromagnetic interaction of the electrons with the surrounding conducting vacuum chamber, and acts back on the bunch perturbing the particle motion[2]. The Fourier transform of the wake-potential is called the *longitudinal coupling impedance*.

Several Authors presented numerical solutions of the PWD equation, with different longitudinal coupling impedances (see[3], and references quoted therein).

Exact solutions have been obtained only for purely resistive[4] and inductive[5] impedances, while approximate solutions have been presented in[6] for a purely capacitive impedance and in[7] for a resonator impedance.

In this paper we propose two techniques (one analytical and one numerical) for solving the PWD equation with a general impedance function. Both techniques are rather general, systematic and computationally efficient (fast and accurate).

The paper is organized as follows. In Section 2 we introduce the PWD equation. In Section 3 we modify the PWD equation to include the normalization condition,

and present a general perturbation scheme for finding a solution in powers of a *small* parameter which is proportional to the bunch particle number; the perturbative solution is then recast into a Pade' approximant, to improve its accuracy, and tested by comparison with known exact analytic (resistive impedance) and numerical (broad-band impedance) solutions. In *Section 4* we present a new algorithm for computing a numerical solution of the Haissinski equation with second-order impedance, which is considerably faster than the usual one[8] (computational burden $\sim N$ as compared to $\sim N^2$ of Bane-Wilson technique, $N$ being the number of computed density values). Representative numerical results are discussed in Section 5. Conclusions are collected under Section 6. explicit formulas pertinent to Sect. 3.2 are reported in Appendix A. A computer code for the symbolic computation of Pade' Approximants, written in REDUCE, is given in Appendix B.

## 2 THE POTENTIAL WELL DISTORTION EQUATION

We consider a bunched electron-beam which travels along the axis of a storage ring, with velocity close to that of light. We assume that the beam has a negligible transverse dimension, and restrict our attention to longitudinal dynamics.

We introduce the well known concept of synchronous particle, which is an ideal particle whose energy is the nominal one, circulating on the ideal orbit. Any other particle can be characterized by the deviations : $\epsilon = E - E_s$, $\tau = (z - z_s)/(\beta c)$, where $E$ and $z$ are respectively the energy and the longitudinal position of the generic particle, the suffix $s$ indicates the synchronous particle, $\beta$ is the relativistic factor and $c$ is the speed of light in *vacuum*. In the synchrotron phase space $\{\tau, \epsilon\}$ the single particle dynamics is described by the following equations[1]:

$$\dot{\tau} = -\frac{\alpha_c \epsilon}{E_s}, \dot{\epsilon} = \frac{eV_{RF}(\tau) - U_0 - D\epsilon}{T_0} - \dot{M}(t), \tag{1}$$

where $\alpha_c$ is the momentum compaction factor : $\alpha_c = (R/R_s - 1)(p/p_s - 1)^{-1}$, $R$ and $p$ being respectively the particle orbital radius and momentum, $V_{RF}(\tau)$ is the accelerating Radio Frequency (RF) potential, $U_0$ is the average energy lost by each particle at every turn, $T_0$ is the synchrotron particle revolution period, $D$ is the damping constant due to radiation, and $\dot{M}(t)$ describes the quantum fluctuations due to photon emission.

The equilibrium distribution function $\psi(\tau, \epsilon)$ in synchrotron phase space for an electron beam is[2]:

$$\psi(\tau, \epsilon) = C \exp\{-\frac{H(\tau, \epsilon)}{H_0}\} = C \exp\{-\frac{\alpha_c \epsilon^2}{2E_s H_0} - \int_0^\tau \frac{eV(\xi) - U_0}{T_0 H_0} d\xi\}, \tag{2}$$

where $H(\tau, \epsilon)$ is the single particle Hamiltonian, $C$ [ $\sec^{-1}$ $\mathrm{erg}^{-1}$] is a constant defined by the normalization condition:

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \psi(\tau, \epsilon) d\epsilon d\tau = N_{tot}, \tag{3}$$

$N_{tot}$ is the total number of beam particles, and:

$$H_0 = \frac{M_m^2 \alpha_c T_0}{2E_s D}, M_m^2 = \lim_{t \to 0} \frac{< M^2(t) >}{t}. \tag{4}$$

A charged particle beam traveling in a conducting accelerator pipe creates a wake field by interacting electromagnetically with its environment. This field acts back on the beam[2] perturbing the particle motion.[a] The action of the wake field on a single particle can be described through the potential:

$$V(\tau) = V_{RF}(\tau) + \int_0^{+\infty} eW(\xi) n(\tau - \xi) d\xi, \tag{5}$$

where $W(\tau)$ is the wake potential [ volt/coulomb ] and $n(\tau)$ [ sec $^{-1}$] is the equilibrium longitudinal particle density:

$$n(\tau) = \int_{-\infty}^{+\infty} \psi(\tau, \epsilon) d\epsilon. \tag{6}$$

The integral in (5) describes the EM potential felt by a single particle in $\tau$ due to all other particles ahead within the same bunch. Note that $W(\tau) = 0$, if $\tau \leq 0$.

Expanding the potential in a power series in $\tau$, assuming (as usual) that upon crossing the RF cavities the bunch gains exactly the same amount of energy lost by radiation at every turn ($eV_{RF}(0) = U_0$), and letting:

$$\sigma = \sqrt{\frac{T_0 H_0}{e\dot{V}_{RF}(0)}} [\text{ sec }], G = \frac{eN_{tot}}{\sqrt{2\pi}\dot{V}_{RF}(0)} [\text{ ohm }^{-1} \text{ sec }^2], K = \frac{2\pi\sigma C}{N_{tot}} \sqrt{\frac{H_0 E_s}{\alpha_c}}, \tag{7}$$

$$i(\tau) = \frac{\sqrt{2\pi}\sigma n(\tau)}{N_{tot}}, \quad s(\tau) = \int_0^{\tau} W(\xi) d\xi [\text{ ohm }], \tag{8}$$

we obtain the Potential Well Distortion equation for the normalized equilibrium longitudinal particle density[7]:

$$i(\tau) = K \exp\{-\frac{\tau^2}{2\sigma^2} - \frac{G}{\sigma^3} \int_0^{+\infty} s(\xi) i(\tau - \xi) d\xi\}, \tag{9}$$

with the normalization condition:

$$\int_{-\infty}^{+\infty} i(\tau) d\tau = \sigma\sqrt{2\pi}. \tag{10}$$

In order to work with dimensionless quantities, we further introduce the variables:

$$x = \omega_0 \tau, y = \omega_0 \xi, \tag{11}$$

[a] The only exception to this state of affairs is the rather artificial case of a lossless conducting pipe with uniform cross-section.

where $\omega_0 = 2\pi/T_0$ and let:

$$\gamma = \sqrt{2}\omega_0\sigma,\ \lambda_0 = \frac{2N_{tot}\omega_0 V_{RF}(0)}{\dot{V}_{RF}(0)},\ \lambda = \frac{\lambda_0}{\gamma^2},\ K' = \frac{K}{\sqrt{\pi}\gamma}, \tag{12}$$

$$u(x) = \frac{i(\omega_0^{-1}x)}{\sqrt{2\pi}\omega_0\sigma},\ S(y) = s(\omega_0^{-1}y)\frac{e\omega_0}{V_{RF}(0)}. \tag{13}$$

The PWD equation (10) thus finally becomes:

$$u(x) = K'\exp\{-\frac{x^2}{\gamma^2} - \lambda\int_0^{+\infty} S(y)u(x-y)dy\}. \tag{14}$$

with the normalization condition:

$$\int_{-\infty}^{+\infty} u(x)dx = 1. \tag{15}$$

In the following we shall always refer to the standard PWD equation (14), together with the normalization condition (15). The latter depend only on two dimensionless parameters, $\gamma$ and $\lambda$ which are related to the (unperturbed) bunch length $\beta\sigma$ and particle number $N_{tot}$, and to the relevant machine parameters. The degree of difficulty in solving equation (14) depends essentially on the choice of $S(y)$, i.e. of the wake potential $W(x)$.

## 3 ANALYTIC SOLUTION FOR A GENERAL IMPEDANCE

Equations (14) and (15) can be combined to give the following integral equation, which we call the modified Haissinski equation:

$$u(x) = \frac{\exp\{-\frac{x^2}{\gamma^2} - \lambda\mathcal{L}(u(x,\lambda))\}}{\{\int_{-\infty}^{+\infty}\exp\{-\frac{x^2}{\gamma^2} - \lambda\mathcal{L}(u(x,\lambda))\}dx\}}, \tag{16}$$

where:

$$\mathcal{L}(u(x,\lambda)) = \int_0^{+\infty} S(y)u(x-y,\lambda)dy, \tag{17}$$

and the potential-well profile is defined as:

$$\Phi(x,\lambda) = \frac{x^2}{\gamma^2} + \lambda\mathcal{L}(u(x,\lambda)). \tag{18}$$

For a fixed $\gamma$, the solution of (16), is a function $u(x,\lambda)$. We assume that it can be expanded in a power series in $\lambda$:

$$u(x,\lambda) = \sum_{n=0}^{+\infty}\lambda^n u^{(n)}(x), \tag{19}$$

where the n-th term is defined as:

$$u^{(n)}(x,\lambda) = \frac{1}{n!}\frac{d^n}{d\lambda^n}u(x)\Big|_{\lambda=0}, \tag{20}$$

and the potential corresponding to (19) is:

$$\Phi_n(x,\lambda) = \frac{x^2}{\gamma^2} + \sum_{j=1}^{n}\lambda^j \mathcal{L}(u^{(j-1)}(x)). \tag{21}$$

Note that:

$$\int_{-\infty}^{\infty} dx u^{(n)}(x) = \delta_{n1}, \tag{22}$$

where $\delta_{n1}$ is the Kronecker function.

Letting (19) into (16), differentiating repeatedly with respect to $\lambda$ and finally letting $\lambda = 0$ one gets a hierarchy of **linear** equations in the unknown functions $u^{(n)}$. The zero-th order term is:

$$u(x,0) = u^{(0)}(x) = \frac{e^{-\frac{x^2}{\gamma^2}}}{\sqrt{\pi}\gamma}, \tag{23}$$

the first order term is:

$$u^{(1)}(x) = u^{(0)}(x)\left\{-\mathcal{L}(u^{(0)}(x)) + \frac{1}{\sqrt{\pi}\gamma}\int_{-\infty}^{+\infty}\mathcal{L}(u^{(0)}(y))e^{-y^2/\gamma^2}dy\right\}, \tag{24}$$

the second order term is:

$$u^{(2)}(x) = \frac{u^{(0)}(x)}{2}\left\{\mathcal{L}^2(u^{(0)}(x)) - 2\frac{\mathcal{L}(u^{(0)}(x)}{\sqrt{\pi}\gamma}\int_{-\infty}^{+\infty}\mathcal{L}(u^{(0)}(y))e^{-y^2/\gamma^2}dy\right.$$

$$+\frac{2}{\pi\gamma^2}\left[\int_{-\infty}^{+\infty}\mathcal{L}(u^{(0)}(y))e^{-y^2/\gamma^2}dy\right]^2 + \frac{2}{\sqrt{\pi}\gamma}\int_{-\infty}^{+\infty}\mathcal{L}(u^{(1)}(y))e^{-y^2/\gamma^2}dy$$

$$\left.-2\mathcal{L}(u^{(1)}(y)) - \frac{1}{\sqrt{\pi}\gamma}\int_{-\infty}^{+\infty}\mathcal{L}^2(u^{(0)}(y))e^{-y^2/\gamma^2}dy\right\}, \tag{25}$$

and so forth.

The hierarchy can be solved systematically, e.g., resorting to algebric manipulation codes (REDUCE, MACSYMA, etc.). However, the formal complexity and computational budget becomes rapidly large with increasing $n$. Unfortunately, the accuracy of the solution obtained by including only a few terms in (19) can be unacceptable if $\lambda \geq 1$. In order to extend its validity, the $N-th$ order truncated solution (19):

$$u_N(x) - u^{(0)}(x) = \sum_{n=1}^{N}\lambda^n u^{(n)}(x), \tag{26}$$

can be transformed into a Pade' approximant:

$$f_{P,Q}(x) = \frac{\sum_{n=1}^{P} \lambda^n \chi_n(x)}{1 + \sum_{m=1}^{Q} \lambda^m \psi_m(x)}, \tag{27}$$

where: $P + Q = N$. Pade' approximants have been recognized as effective *analytic continuation* tools, suitable to extend the validity of perturbation solutions in several branches of applied physics[9].

The functions $\chi_n$, $n = 1, 2, ..., P$, $\psi_m$ $n = 1, 2, ..., Q$ can be computed by requiring that:

$$f_{PQ}(x) - \left(u_{P+Q}(x) - u^{(0)}(x)\right) = \mathcal{O}(\lambda^{P+Q+1}), \tag{28}$$

which gives a straightforward system of linear equations, in the unknown functions: $\chi_m(x), m = 1, 2, ..., P$, and $\psi_n(x), n = 1, 2, ..., Q$:

$$u^{(s)}(x) + \sum_{m=1}^{\min(P,s-1)} u^{(s-m)}(x)\psi^{(m)}(x) = \begin{cases} \chi^{(s)}(x) & 1 \le s \le Q \\ 0 & Q+1 \le s \le P+Q. \end{cases} \tag{29}$$

### *3.1* Resistive Impedance

In this section we check the accuracy of the proposed analytical solution of equation (14–15) by comparison with the *exact* one obtained in[4] for the special case of a longitudinal resistive impedance:

$$W(\tau) = R_s \delta(\tau), \tag{30}$$

for which the Haissinski equation (14) can be written:[b]

$$u(x) = K' \exp\left\{-\frac{x^2}{\gamma^2} - \Lambda \int_0^x u(y)dy\right\}, \tag{31}$$

[b] This can be most readily seen by first using (30) into (5) then eq. (2), and finally eq.s (7), the first of (8), (11), (12) and the first of (13).

where:

$$\Lambda = \frac{e\omega_0 R_s}{V_{RF}(0)}\lambda. \tag{32}$$

The solution:

$$u(x,\Lambda) = \sum_{n=0}^{+\infty} \Lambda^n u^{(n)}(x), \tag{33}$$

is obtained with relative ease [10], and reads:

$$u^{(0)}(x) = \frac{\exp\left(-x^2/\gamma^2\right)}{\gamma\sqrt{\pi}}, \tag{34}$$

$$u^{(1)}(x) = \frac{\exp\left(-x^2/\gamma^2\right)}{2\gamma\sqrt{\pi}} \operatorname{erf}(x/\gamma), \tag{35}$$

$$u^{(2)}(x) = \frac{\exp\left(-x^2/\gamma^2\right)}{2^2\gamma\sqrt{\pi}} \left[\operatorname{erf}^2(x/\gamma) - \frac{1}{3}\right], \tag{36}$$

$$u^{(3)}(x) = \frac{\exp\left(-x^2/\gamma^2\right)}{2^3\gamma\sqrt{\pi}} \left[\operatorname{erf}^3(x/\gamma) - \frac{2}{3}\operatorname{erf}(x/\gamma)\right], \tag{37}$$

$$u^{(4)}(x) = \frac{\exp\left(-x^2/\gamma^2\right)}{2^4\gamma\sqrt{\pi}} \left[\operatorname{erf}^4(x/\gamma) - \operatorname{erf}^2(x/\gamma) + \frac{2}{15}\right], \tag{38}$$

$$u^{(5)}(x) = \frac{\exp\left(-x^2/\gamma^2\right)}{2^5\gamma\sqrt{\pi}} \left[\operatorname{erf}^5(x/\gamma) - \frac{4}{3}\operatorname{erf}^3(x/\gamma) + \frac{17}{45}\operatorname{erf}(x/\gamma)\right], \tag{39}$$

$$u^{(6)}(x) = \frac{\exp\left(-x^2/\gamma^2\right)}{2^6\gamma\sqrt{\pi}} \left[\operatorname{erf}^6(x/\gamma) - \frac{5}{3}\operatorname{erf}^4(x/\gamma) + \frac{11}{15}\operatorname{erf}^2(x/\gamma) - \frac{17}{315}\right], \tag{40}$$

$$u^{(7)}(x) = \frac{\exp\left(-x^2/\gamma^2\right)}{2^7\gamma\sqrt{\pi}} \left[\operatorname{erf}^7(x/\gamma) - 2\operatorname{erf}^5(x/\gamma) + \right.$$

$$\left. + \frac{6}{5}\operatorname{erf}^3(x/\gamma) - \frac{62}{315}\operatorname{erf}(x/\gamma)\right], \tag{41}$$

$$u^{(8)}(x) = \frac{\exp\left(-x^2/\gamma^2\right)}{2^8\gamma\sqrt{\pi}} \left[\operatorname{erf}^8(x/\gamma) - \frac{7}{3}\operatorname{erf}^6(x/\gamma) + \right.$$

$$\left. + \frac{16}{9}\operatorname{erf}^4(x/\gamma) - \frac{88}{189}\operatorname{erf}^2(x/\gamma) + \frac{62}{2835}\right], \tag{42}$$

*etc.*

The [2, 2] and [4, 4] Pade' approximants built from (34–42) are, respectively:

$$u_{[2,2]}(x) = \frac{\exp(-x^2/\gamma^2)}{\gamma\sqrt{\pi}} \frac{(\Lambda/2)^2 + 3(\Lambda/2)\,\mathrm{erf}\,(x/\gamma) + 15}{(\Lambda/2)^2[3\,\mathrm{erf}^2(x/\gamma) - 6] + 12(\Lambda/2)\,\mathrm{erf}\,(x/\gamma) - 15}, \quad (43)$$

and:

$$u_{[4,4]}(x) = \frac{\exp(-x^2/\gamma^2)}{\gamma\sqrt{\pi}} \Big[ (\Lambda/2)^4 + 10(\Lambda/2)^3\,\mathrm{erf}\,(x/\gamma) +$$

$$+ 105(\Lambda/2)^2 + 105(\Lambda/2)\,\mathrm{erf}\,(x/\gamma) + 945\Big] \cdot$$

$$\cdot \Big\{ (\Lambda/2)^4[15 - 10\,\mathrm{erf}^2(x/\gamma)] - 60(\Lambda/2)^3\,\mathrm{erf}\,(x/\gamma) +$$

$$+ (\Lambda/2)^2[420 - 105\,\mathrm{erf}^2(x/\gamma)] - 840(\Lambda/2)\,\mathrm{erf}\,(x/\gamma) + 945\Big\}^{-1}. \quad (44)$$

The comparison between the *plain* series solution (34–38), its [2, 2] and [4, 4] Pade' counterparts (43,44) and the exact solution obtained in[4]:

$$u(x) = \frac{1}{\gamma\sqrt{\pi}} \frac{\exp(-x^2/\gamma^2)}{(\Lambda/2)[\coth(\Lambda/2) - \mathrm{erf}\,(x/\gamma)]} \quad (45)$$

is discussed in Sect. 5.

### *3.2* Broad Band Impedance

For each cavity mode of oscillation, it is possible to represent $Z(\omega)$, as a resonant (parallel) circuit[5]:

$$Z(\omega) = \frac{R_s}{1 + iQ[(\omega/\omega_0) - (\omega_0/\omega)]}, \quad (46)$$

$\omega_0$ being the angular resonant frequency, $Q$ the quality factor, and $R_s$ the shunt resistance.

Letting $Q = 1/2$ (critical damping) we easily get:

$$s(\tau) = 2R_s\omega_0 H(\tau)\tau e^{-\omega_0\tau}, \quad (47)$$

where $H(\tau)$ is the Heaviside function. The modified PWD equation (16) accordingly reads:

$$u(x) = \frac{\exp\{-\frac{x^2}{\gamma^2} - \Lambda\int_0^{+\infty} ye^{-y}u(x-y)dy\}}{\int_{-\infty}^{+\infty} \exp\{-\frac{x^2}{\gamma^2} - \Lambda\int_0^{+\infty} ye^{-y}u(x-y)dy\}dx}. \quad (48)$$

After long and tedious calculations we obtain the second order perturbative solution of (48) reported in Appendix A.

This solution, together with the [0, 2] Pade' approximant obtained thereof:

$$u_{[0,2]} = \frac{u^{(0)}(x)}{1 - \Lambda \frac{u^{(1)}(x)}{u^{(0)}(x)} + \Lambda^2 \left[ \left( \frac{u^{(1)}(x)}{u^{(0)}(x)} \right)^2 - \frac{u^{(2)}(x)}{u^{(0)}(x)} \right]}, \tag{49}$$

has been compared to the standard numerical solution. The results are discussed in Sect. 5.

## 4 NUMERICAL SOLUTION FOR A GENERAL IMPEDANCE

The standard numerical method for solving (14) goes through the computation of $N$ density values, $u_k = u(-x_0 + k\Delta x)$, $k = 0, 1, ...N - 1$, $x_0$ being chosen in such a way that:

$$u(-x_0) \sim \exp\left\{ -\frac{x_0^2}{\gamma^2} \right\}. \tag{50}$$

Equation (14) can be used to compute the value of $u_k$ by using only the *known* values of $u_1, u_2, ..., u_{k-1}$ for calculating the integral on the r.h.s., in view of the fact that $S(0) = 0$.

The computation of $u_k$ requires accordingly a number of multiplications $\sim k$. Hence, if $N$ is the total number of time steps, one needs about $N^2$ floating-points operations for computing $u(x)$, *for each trial value* of the normalization constant $K$.

We propose a different numerical technique for solving (14)[10] which consists in *first* transforming it into an equivalent nonlinear ordinary differential equation, and *then* solving this latter, using a standard (e.g., Runge-Kutta) algorithm. This provides a *substantial* improvement in speed, because the total number of floating point operations needed for computing $u(x)$, for each trial value of $K$ is now of the order of $N$, as will be shown soon.

The integral equation (14) can be trasformed into an equivalent nonlinear differential equation as follows. We first let:

$$\frac{u(x)}{K} = e^{V(x)}, \tag{51}$$

whence:

$$V(x) = -\frac{x^2}{\gamma^2} - \lambda K \int_0^{+\infty} S(y) e^{V(x-y)} dy, \tag{52}$$

with the normalization condition:

$$\int_{-\infty}^{+\infty} e^{V(x)} dx = \frac{1}{K}. \tag{53}$$

As the longitudinal dimension of the bunch is very small compared to the machine circumference, and due to the fact that we neglect the interaction among different bunches, we assume: $S(+\infty) = \dot{S}(+\infty) = 0$. Further whenever $S(y)$ satisfies the equation:[c]

$$\ddot{S}(x) = a_1 \dot{S}(x) + a_2 S(x), \tag{54}$$

with the initial conditions $S(0) = 0$, $\dot{S}(0) = S_{10}$, then differentiating twice (52), we get the following ordinary nonlinear differential equation for $V(x)$:

$$\ddot{V}(x) - a_1 \dot{V}(x) - a_2 V(x) = a_2 \frac{x^2}{\gamma^2} + \frac{2a_1 x}{\gamma^2} - \frac{2}{\gamma^2} - \lambda \dot{S}(0) K e^{V(x)}, \tag{55}$$

whose parameters are $\lambda$, $\gamma$, $a_1$, $a_2$ and $S_{10}$. Equation (55) can be integrated starting from an $x_0$ such that:

$$V(x_0) \sim -\frac{x_0^2}{\gamma^2}, \quad \dot{V}(x_0) \sim -\frac{2x_0}{\gamma^2}, \tag{56}$$

consistent with (52).

For each trial value of $K$, we can use, e.g., a Runge-Kutta algorithm for solving equation (55), with a computational burden of $\sim N$ floating point operations, $N$ being the number of computed $V$ values.

For the special case of a broadband impedance ($S_{10} = 1$, $a_1 = -2$ and $a_2 = -1$), where (55) becomes:

$$\ddot{V}(x) + 2\dot{V}(x) + V(x) = -\frac{x^2}{\gamma^2} - \frac{4x}{\gamma^2} - \frac{2}{\gamma^2} - \Lambda K e^{V(x)}, \tag{57}$$

we compared our solution to the one obtained by using the standard (Bane-Wilson) technique. The results agree to all decimal figures in single computer precision (see also Sect. 5).

## 5 COMPUTATIONAL RESULTS

In this section we give a number of computational results, and comment upon the comparison between our analytical and numerical solutions of the PWD equation and the available ones.

[c] This is the case for a resonant cavity. In the language of system theory equation (54) describes second order systems, whose transfer functions have two poles. The whole procedure is easily generalized, in principle, to arbitrary impedance functions.

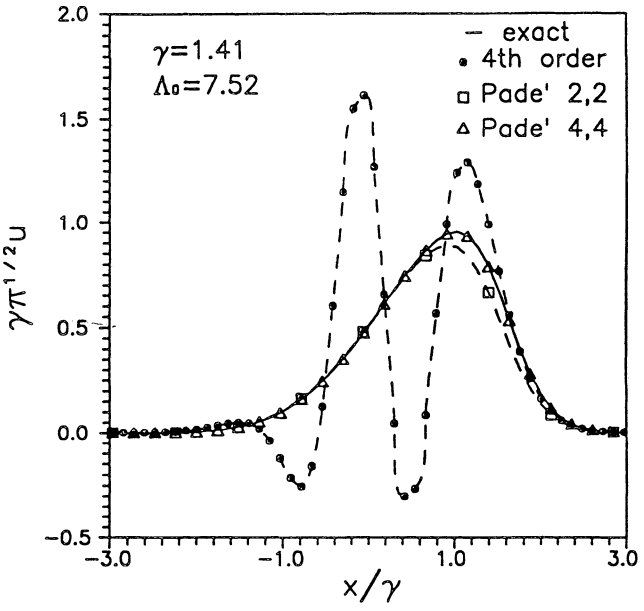


FIGURE 1: Longitudinal Bunch Density for Pure Resistive Impedance with $\gamma$=1.41 and $\Lambda_0$=7.52 (– exact, • 4th order, □ Pade' [2,2], △ Pade' [4,4]).

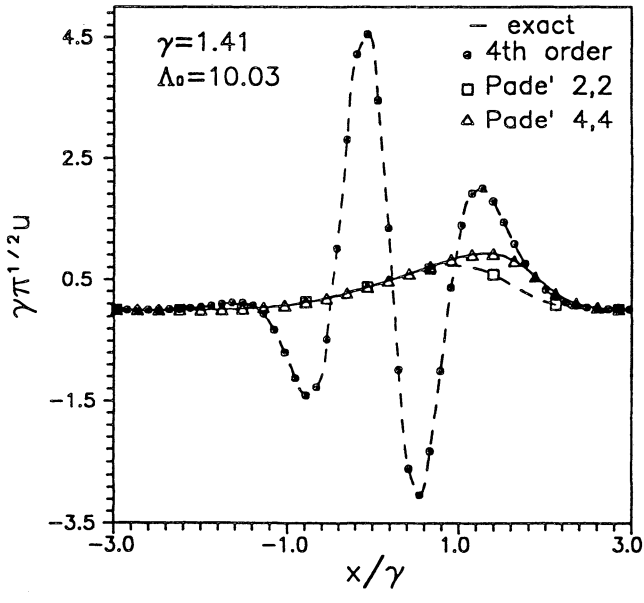


FIGURE 2: Longitudinal Bunch Density for Pure Resistive Impedance with $\gamma$=1.41 and $\Lambda_0$=10.03 (– exact, • 4th order, □ Pade' [2,2], △ Pade' [4,4]).

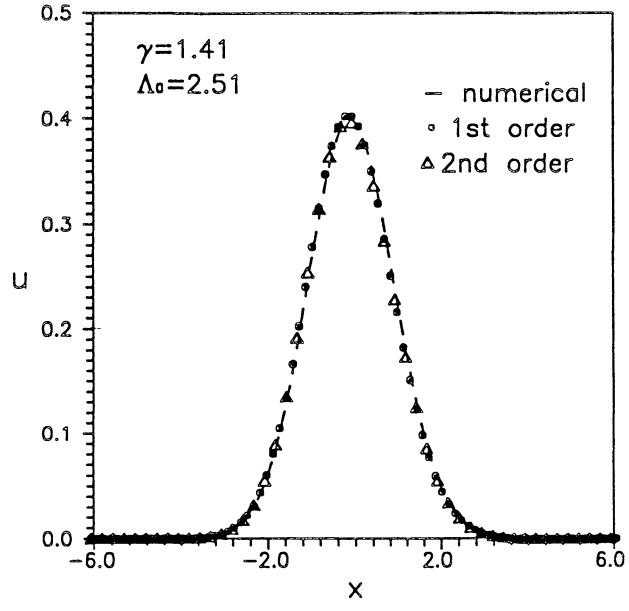


FIGURE 3(a):

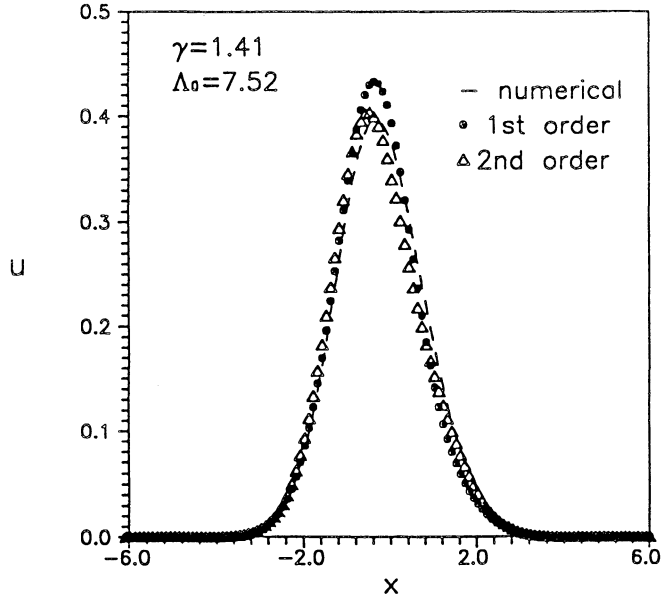


FIGURE 3(b):

FIGURE 3(a),(b): Longitudinal Bunch Density for Broad Band Impedance with $\gamma=1.41$ and $\Lambda_0=2.51,7.52$ (– numerical, • 1st order, $\Delta$ 2nd order).

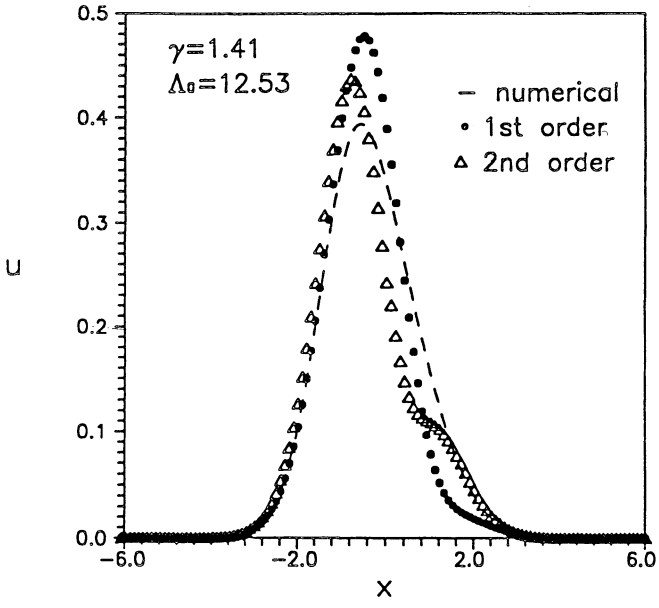


FIGURE 3(c):

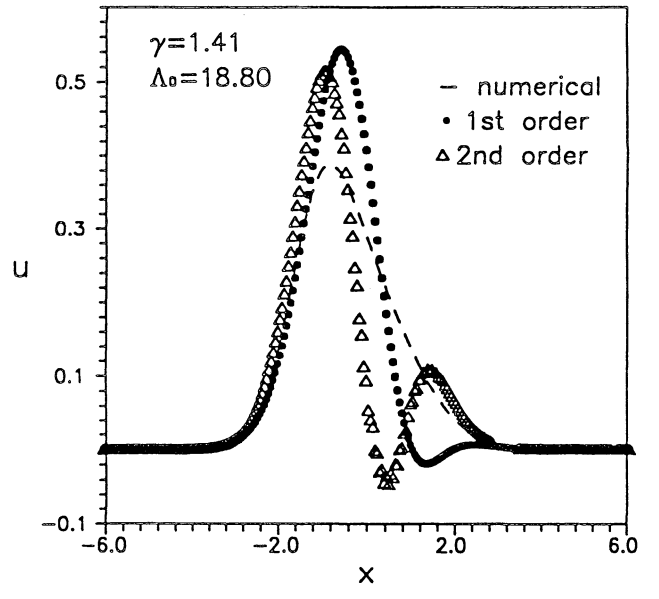


FIGURE 3(d):

FIGURE 3(c),(d): Longitudinal Bunch Density for Broad Band Impedance with $\gamma=1.41$ and $\Lambda_0=12.53, 18.80$ (– numerical, • 1st order, △ 2nd order).

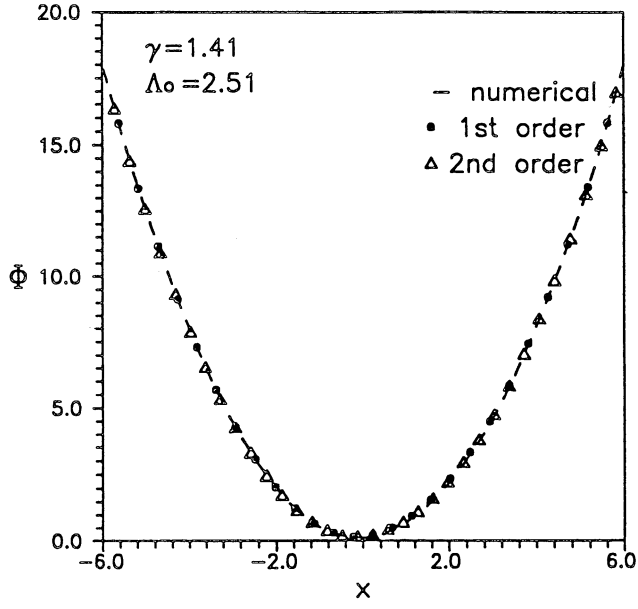


FIGURE 4(a):

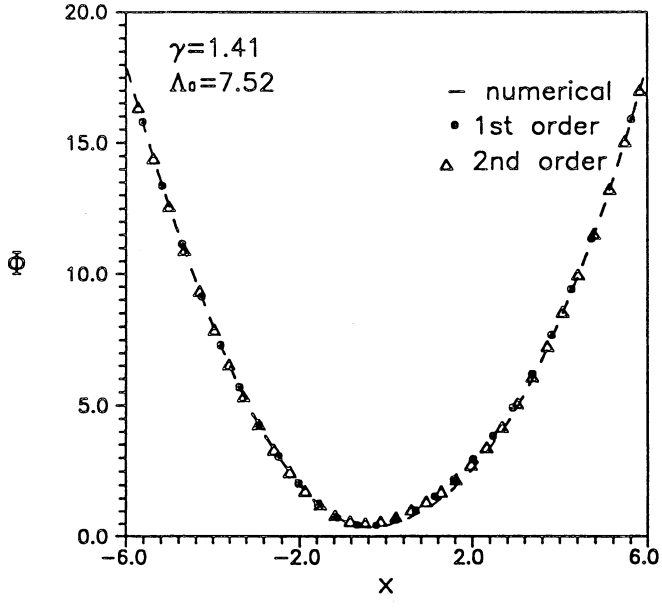


FIGURE 4(b):

FIGURE 4(a),(b): Longitudinal Potential Well for Broad Band Impedance with $\gamma$=1.41 and $\Lambda_0$=2.51,7.52 (– numerical, • 1st order, Δ 2nd order).

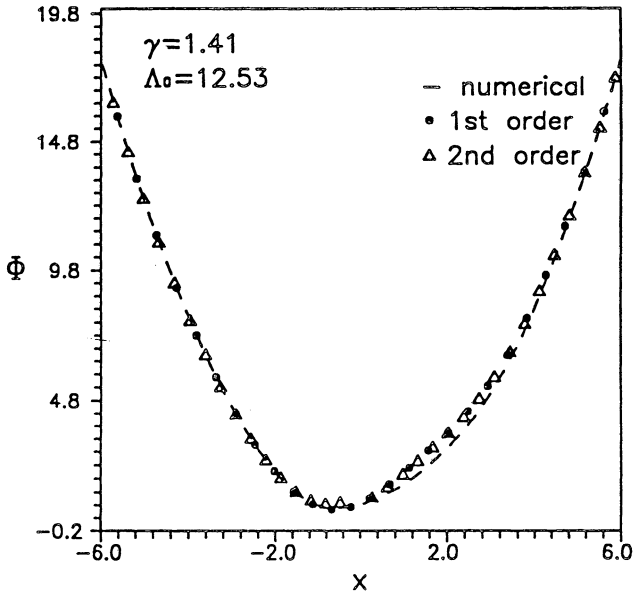


FIGURE 4(c):

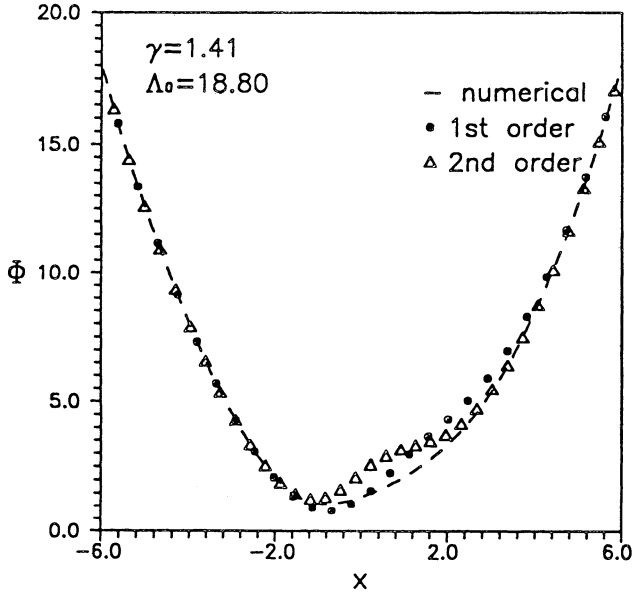


FIGURE 4(d):

FIGURE 4(c),(d): Longitudinal Potential Well for Broad Band Impedance with $\gamma=1.41$ and $\Lambda_0=12.53,18.80$ (– numerical, • 1st order, △ 2nd order).

A comparison between the available exact solution for resistive impedance[4], our $4-th$ order analytic solution (34–38), and its $[2,2]$ and $[4,4]$ Pade' counterparts is shown in Fig.s 1,2, which clearly exhibits the dramatic improvement in accuracy obtained by the Pade approximant.

In Fig.s 3,4 we plot, for different values of the parameters $\Lambda_0$ and $\gamma$, the normalized bunch charge density $u(x)$ and the potential well $\Phi(x)$ for a critically damped broadband impedance[d].

The following expected common features are easily noted: it is seen that by increasing $\Lambda_0$, (i.e. the total bunch charge) the bunch becomes initially narrower and the peak density increases, until some critical value of $\Lambda_0$ is reached. Further increasing $\Lambda_0$, the peak density decreases, the bunch becomes wider, and its baricenter is shifted behind, because of energy loss. Correspondingly, it is seen from Fig. 4 that by increasing the bunch charge, the parabolic potential well is increasingly distorted by the wake-field, thus reducing the well-depth and hence the stability of the bunch.

For $\gamma = 1.41$ our 2-nd order perturbative solution is accurate up to $\Lambda_0 = 7.52$ and, as we should expect, is more accurate than the 1-st order one. Both, however become poor at larger values of $\Lambda_0$. From Fig. 5, we can see that the $[0,2]$ Pade' (49) approximant obtained from the 2-nd order solution behaves significantly better. even at $\Lambda_0 = 18.8$, which is a pretty large value.

## 6 CONCLUSIONS

We presented two different methods (one analytical and one numerical) for solving the PWD equation, which plays an important role in Particles Accelerators. The analytical method is general, systematic, and significantly more accurate than off the shelf perturbative methods.

The numerical method relies on the possibility of reformulating the nonlinear Haissinski integral equation into a nonlinear differential equation, and leads to a significant reduction of computational budget.

Using the above tools in a combined strategy would allow a systematic study of the bunch dynamics in the parameter-space, within the limits of the PWD model. It could also prove interesting to study the *inverse* problem related to eq. (16), i.e., by solving for the unknown impedance (wake) function, so as to obtain a specified bunch profile.

---

[d] These curves have been computed for $2R_s\omega_0 = 90$ ohm ps$^{-1}$; $\omega = 1/\tau_r = 1/120$ ps$^{-1}$. The values $\sigma$=75(120) ps correspond to (see eq. 12): $\gamma$=.88(1.41), while the values $G$=40(80,120,160,200) ps$^2$ ohm$^{-1}$ therein correspond to (see eq. 32): $\Lambda_0$=2.51 (5.01,7.52,10.03,12.53). The values of $\sigma$ and $G$ are those used in [8]. However, some of the parameters values considered are not attainable in real machines [8]. For example, in SPEAR the values $\sigma$=75 ps and $G$>100 ps$^2$ ohm$^{-1}$ correspond to a current above the turbulence threshold.

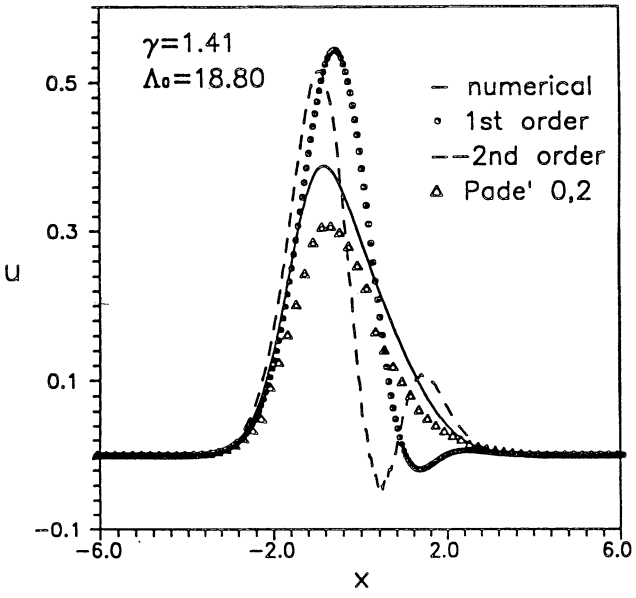


FIGURE 5: Longitudinal Bunch Density for Broad Band Impedance with $\gamma$=1.41 and $\Lambda_0$=18.80 ( – numerical, • 1st order, – – 2nd order, Δ Pade' [0,2]).

## ACKNOWLEDGEMENTS

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APPENDIX A - BROADBAND IMPEDANCE:

We give here the formulas for the first and second order terms of the solution of (48) in powers of $\Lambda$. The first order term is the following:

$$u^{(1)}(x) = \frac{e^{-x^2/\gamma^2}}{\pi\gamma^2}\left\{\frac{L_0\gamma}{2\sqrt{\pi}}e^{\gamma^2/4} - L[u^{(0)}(x)]\right\},$$

with:

$$L[u^{(0)}(x)] = \frac{\gamma^2}{2}e^{\gamma^2/4-x}\left\{e^{-(\gamma/2-x/\gamma)^2} - \sqrt{\pi}(\frac{\gamma}{2}-\frac{x}{\gamma})\left[1-\ \mathrm{erf}\,(\frac{\gamma}{2}-\frac{x}{\gamma})\right]\right\},$$

$$L_0 = \sqrt{\pi}\gamma\left\{\frac{1}{\sqrt{2}}e^{-\gamma^2/4} + e^{\gamma^2/4}\left[\frac{1}{\sqrt{2}} - \gamma\sqrt{\pi} + \gamma\sqrt{\pi}\ \mathrm{erf}\,(\frac{\gamma}{\sqrt{2}})\right]\right\},$$

where erf $(\cdot)$ is the error function. In order to write the second order term we let:

$$L[u^{(o)}(x)] = \frac{1}{\pi\gamma^2}\left[\frac{L_0\gamma}{2\sqrt{\pi}}e^{\gamma^2/4}L[u^{(0)}(x)] - T(x)\right],$$

where:

$$T(x) = \frac{\gamma^3}{2}\left\{\left[1+\ \mathrm{erf}\left(\frac{\sqrt{2}x}{\gamma}-\frac{\gamma}{2\sqrt{2}}\right)\right]e^{-x+\gamma^2/8}\sqrt{\frac{\pi}{2}}x + \frac{\gamma}{4}e^{-2x^2/\gamma^2}\right.$$

$$-\left[1+\ \mathrm{erf}\,(\frac{x}{\gamma})\right]e^{-x+\gamma^2/4}\frac{\pi}{4}\gamma(1+x) + \frac{\gamma^2}{4}\sqrt{\pi}e^{-x-x^2/\gamma^2+\gamma^2/4}\ \mathrm{erf}\,(\frac{\gamma}{2}-\frac{x}{\gamma})+$$

$$\left.+\sqrt{\pi}e^{-x}\frac{\gamma}{2}\left[(1+x)\int_{\frac{\gamma}{2}-\frac{x}{\gamma}}^{+\infty}dye^{-y^2+\gamma y}\ \mathrm{erf}\,(y) - \left(\gamma e^{\gamma^2/4} - \frac{e^{-x^2/\gamma^2+x}}{\sqrt{\pi}}\right)\frac{e^{-x^2/\gamma^2}}{2}\right]\right\}.$$

We let further:

$$L_1 = \frac{1}{\gamma\sqrt{\pi}}\left[\frac{L_0\gamma}{2\sqrt{\pi}}e^{\gamma^2/4}A_1 - T_1\right],$$

where:

$$A_1 = \frac{\gamma^3\sqrt{\pi}}{2}\left\{\frac{1}{\sqrt{2}} - \sqrt{\pi}\gamma e^{\gamma^2/2}\left[1-\ \mathrm{erf}\,(\frac{\gamma}{\sqrt{2}})\right]\right\},$$

and:

$$T_1 = \frac{\sqrt{\pi}}{2}\gamma^3 \left\{ \frac{B}{\sqrt{2}} e^{\gamma^2/8} - \frac{\gamma\sqrt{\pi}}{4} e^{\gamma^2/4} C + \frac{\sqrt{\pi}}{4\sqrt{2}} e^{3\gamma^2/8} \left[ \text{erf} \left( \frac{\gamma\sqrt{3}}{2\sqrt{2}} \right) - \gamma^3 \right] + \frac{\gamma^2}{2\sqrt{3}} \right.$$

$$\left. + \frac{\gamma}{2} \int_{-\infty}^{+\infty} dx(1+x) e^{-x-x^2/\gamma^2} \int_{\frac{\gamma}{2}-\frac{x}{\gamma}}^{+\infty} dy e^{-y^2+\gamma y} \text{erf}\,(y) \right\},$$

where:

$$B = \frac{\gamma}{2} \left\{ \gamma e^{-\gamma^2/8} \left[ 2\sqrt{\frac{2}{3}} + \gamma\sqrt{\pi} e^{3\gamma^2/8} \text{erf} \left( \frac{\gamma\sqrt{3}}{2\sqrt{2}} \right) \right] - \sqrt{\pi} e^{\gamma^2/4} \right\},$$

$$C = \gamma e^{\gamma^2/8} \left\{ \sqrt{\pi} e^{\gamma^2/8} \left[ \text{erf} \left( \frac{\gamma}{2\sqrt{2}} \right) \left( \frac{\gamma^2}{2} - 1 \right) + \frac{1}{2} \right] + \frac{\gamma}{\sqrt{2}} \right\}.$$

Finally we let:

$$L_2 = [A_2 + T_2]\gamma^3 \frac{e^{\gamma^2/2}}{4\sqrt{\pi}},$$

with:

$$A_2 = \sqrt{\pi} \left( 1 + \frac{9\gamma^2}{2} \right) \left[ \frac{e^{-5\gamma^2/4}}{2} \int_{-\infty}^{+\infty} dx\, \text{erf}^{\,2}(x) e^{-x^2+3\gamma x} - \right.$$

$$\left. - \sqrt{\pi} e^{\gamma^2} \text{erf} \left( \frac{3\gamma}{2\sqrt{2}} \right) + \frac{\sqrt{\pi} e^{\gamma^2}}{2} \right],$$

$$T_2 = 3\gamma\sqrt{\frac{\pi}{2}} e^{-\gamma^2/8} \left[ \frac{1}{2}(3 + e^{-3\gamma^2/8}) \text{erf} \left( \frac{\sqrt{3}\gamma}{2\sqrt{2}} \right) - 2 \right] + \left( 2 + \frac{1}{\sqrt{3}} \right) \frac{e^{-\gamma^2/2}}{\sqrt{3}}.$$

Summing up, we can write the second order term as follows:

$$u^{(2)}(x) = \frac{u^{(0)}(x)}{2} \left\{ L^2[u^{(0)}(x)] - 2L[u^1(x)] - L[u^{(0)}(x)] \frac{L_0}{\pi} e^{\gamma^2/4} + \right.$$

$$\left. + \frac{2L_1 - L_2}{\gamma\sqrt{\pi}} + e^{\gamma^2/2} \frac{L_0^2}{2\pi^2} \right) \right\}.$$

## APPENDIX B - REDUCE CODE FOR PADE APPROXIMANTS.

```
% ......................
% Program POWTOPAD.RED
% Symbolic Computation of PA
% by Stefania Petracca,
% Read File POWSER containing
% Coefficients of Original
% Truncated Analytic Element
IN "POWSER.RED";
%
% Create 1st Pade' System
% ROW(I)=i-th Row in System
%
B(0):=1;
FOR S:=NUP+1:NUP+NDN DO
BEGIN
NT:=MIN(NDN,S);
ROW(S):=FOR K:=0:NT
SUM B(K)*C(S-K);
END;
%
% Create 2nd Pade' System
%
FOR S:=0:NUP DO
BEGIN
NT:=MIN(NDN,S);
RROW:=FOR K:=0:NT
SUM B(K)*C(S-K);
ROW(S):=RROW-A(S);
END;
%
% Dump Pade' System
% to File PASYS.
%
OFF ECHO;
OUT PASYS;
WRITE "SOLVE(LST(";
FOR K:=0:NUP+NDN-1
DO WRITE ROW(K),",";
WRITE ROW(NUP+NDN),"),";
NTY:=4;
NUP:=2;
NDN:=2;
%
C(0):=C0;
FOR K:=0:NUP
DO WRITE A(K),",";
FOR K:=1:NDN-1
DO WRITE B(K),",";
WRITE B(NDN),");";
WRITE ";END;";
SHUT PASYS;
ON ECHO;
% Solve Pade' System
%
IN "PASYS";
% Dump PA Coefficients
% to File PADOUT
% (FORTRAN style)
%
OFF ECHO;
ON FORT;
OUT PADOUT;
FOR K:=0:NUP
DO WRITE "A(",K,")=",
SOLN(1,K+1);
WRITE "B(0)=",B(0);
FOR K:=1:NDN
DO WRITE "B(",K,")=",
SOLN(1,NUP+K+1);
SHUT PADOUT;
OFF FORT;
ON ECHO;
;END;
```

```
% ......................
% File POWSER.RED
% (Sample Input File
% for use with POWTOPAD.RED)
% by Stefania Petracca,
%
% C_0,...,C_NTY : coeffts
% of x^0,...,x^NTY
% in the Original VS
%
% NUP Order of PA Numerator
% NDN Order of PA Denominator
% (MUST be NUP+NDN=NTY)
C(1):=C1;
C(2):=C2;
C(3):=C3;
C(4):=C4;
;END;
```